MATH 3235 Probability Theory

11/03/22

$$M_X(t) = \mathbb{E}(e^{tX})$$

$$M_X^{(n)}(0) = \mathbb{E}(X^n)$$

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

$$Z = aX + b$$

$$M_Z(t) = \mathbb{E}\left(e^{(aX+b)t}\right) =$$

$$= e^{bt}\mathbb{E}\left(e^{taX}\right) = e^{bt} M_X(at)$$

if Z is $N(0,1)$

$X = \sigma Z + \mu \sim N(\mu, \sigma^2)$

$$M_X(t) = e^{\mu t} e^{\frac{1}{2}\sigma t^2} = e^{\mu t + \frac{1}{2}\sigma t^2}$$

If X is $N(\mu, \sigma^2) \implies$

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t}$$

$$\phi_X(t) = \mathbb{E}\left(e^{itX}\right) = M_X(it)$$

$$= \mathbb{E}(\cos tX) + i\,\mathbb{E}(\sin tX)$$

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f_X(x)\,dx$$

"almost" Fourier Transform of f_X

ϕ_X is a complex number

$$|\phi_X(t)| = \left| \int_{-\infty}^{\infty} e^{itx} f_X(x)\,dx \right| \leq$$

$$\leq \int_{-\infty}^{\infty} \left| e^{itx} f_X(x)\,dx \right| \leq$$

$$\leq \int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

$\phi_X(t)$ exists for every t and

$|\phi_X(t)| \leq 1$. $\phi_X(0) = 1$

$$\frac{d}{dt} \mathbb{E}\left(e^{itX}\right) = \frac{d}{dt} \mathbb{E}\left(e^{itX}\right) \text{"=" } \mathbb{E}\left(\frac{d}{dt} e^{itX}\right) =$$

$$\mathbb{E}\left(iX e^{itX}\right)$$

$$\frac{d}{dt} \phi_X(0) = i\, \mathbb{E}(X)$$

$$\frac{d^n}{dt^n} \phi_X(0) = i^n\, \mathbb{E}(X^n)$$

$$\phi_{X+Y}(t) = \phi_X(t)\, \phi_Y(t)$$

$$\phi_{aX+b}(t) = e^{bit}\,\phi(at)$$

Cauchy r.v. X

$$f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}$$

$$M_X(t) = \begin{cases} 1 & t = 0 \\ +\infty & t \neq 0 \end{cases}$$

$$\phi_X(t) = e^{-|t|}$$

$$\int \frac{\cos tx}{1+x^2}\,dx$$

$$\phi_X(0) = 1$$

$\frac{d}{dx}\phi_X(0)$ is not defined

if Z is $N(0,1)$ Then

$$M_X(t) = e^{\frac{t^2}{2}}$$

$$\phi_X(t) = M_X(it) = e^{-\frac{t^2}{2}}$$

Inverse characteristic Transform

$\phi_X(t) = \mathbb{E}(e^{itX})$ Then

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\,\phi_X(t)\,dt$$

Formal!

$M_X(t)$ is m.g.f. of X

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\,M_X(it)\,dt$$

$X_1 \quad X_2$ are $N(0,1)$

$$M_{X_1}(t) = M_{X_2}(t) = e^{\frac{t^2}{2}}$$

$$M_{X_1+X_2}(t) = e^{\frac{t^2}{2}} e^{\frac{t^2}{2}} = e^{t^2}$$

if X is $N(0,2)$

$$M_X(t) = e^{t^2}$$

$X_1 + X_2$ is $N(0,2)$

$X_1 \quad X_2$ are Cauchy r.v.

$$X_1 + X_2 = Z$$

$$f_Z(z) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{1}{1+(z-x)^2} \frac{1}{1+x^2} \, dx$$

$$\phi_{X_1}(t) = \phi_{X_2}(t) = e^{-|t|}$$

$$\phi_Z(t) = e^{-2|t|}$$

$X_1 + X_2$ has the same p.d.f. of

$2X_1$

if X_1 and X_2 are $N(0,1)$

Than $X_1 + X_2$ has the same

p.d.f. of $\sqrt{2}\, X_1$

0

Review for Midterm 2.

Definition of continuous r.v.

and basic propoerties.

f_X p.d.f.

F c.d.f. (d.f.)

Main Ex.

Exponential r.v. :

$$f_X(x) = \lambda e^{-\lambda x}$$

X is exp par λ

$$\mathbb{E}(X) = \frac{1}{\lambda}$$

$$f_X(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}$$

$$\mathbb{E}(X) = \mu$$

Uniform in [A, B]

$$f_X(x) = \begin{cases} \frac{1}{B-A} & A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$

Normal r.v. $N(\mu, \sigma^2)$

$$f_X = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x-\mu}{2\sigma^2}}$$

if X is $N(0, 1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\phi(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

Both for 1 or 2 r.v.

Change of Variable formula!

Function of a continuos r.v.

Expectations.

X is a r.v.

$Y = h(X)$ Then

$$\mathbb{E}(Y) = \int h(x) f_X(x)\, dx$$

X_1 and X_2

marginals

$Y = h(X_1, X_2)$

$$\mathbb{E}(Y) = \int h(x_1, x_2) f_X(x_1, x_2)\, dx_1 dx_2$$

Independence.